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HARMONIC INSTABILITY OF THE FREE SURFACE OF A LOW-VISCOSITY LIQUID IN A VERTICALLY OSCILLATING VESSEL†

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The linear problem of the parametric excitation of three-dimensional standing waves on the free surface of a liquid of low viscosity in a vessel of arbitrary shape, undergoing vertical oscillations, is investigated. The so-called harmonic instability, for which the natural frequency of the excited waves is close to the oscillation frequency of the vessel, is considered. Using the idea of a boundary layer and the Krylov-Bogolyubov averaging method, approximate formulae are derived for the quantities which define the conditions for harmonic instability—for the threshold oscillation amplitude of the vessel and the limits of the resonance zones. It is shown experimentally that it is possible for harmonic instability to occur on the surface of water in a rectangular vessel. The calculated values of the threshold amplitude and the limits of the resonance zones agree well with those measured experimentally.

When a layer of a heavy liquid oscillates in a vertical direction so-called sub-harmonic instability usually occurs, when the natural frequency of the excited waves is close to half the oscillation frequency of the vessel. This instability has been investigated fairly well both for an ideal liquid and for a viscous liquid [1–3]. Harmonic instability has been investigated to a lesser extent. The possibility of the occurrence of such instability in a vessel of infinitely large horizontal dimensions was investigated in [4], and it was shown that harmonic instability can only occur in a fairly shallow viscous liquid.

In this paper we derive the conditions for harmonic instability of the free surface of a liquid of low viscosity in a vessel of finite horizontal dimensions and we check these conditions experimentally. To solve the problem the velocity field of the liquid is divided into potential and eddy components and the boundary-layer method is employed. The effectiveness of this approach was demonstrated for the first time in [5,6] when investigating a low-viscosity liquid. By analogy with a previous paper [3] we also use the idea of the Krylov–Bogolyubov averaging method.

1. FORMULATION OF THE PROBLEM

A vessel containing a viscous incompressible liquid with a free surface undergoes vertical oscillations as given by $-s\cos\Omega t$. The free surface is a horizontal plane in the equilibrium state. We will assume that, at the instant $t = 0$, a perturbation occurs in the liquid in the form of a standing wave with an infinitely low amplitude and a frequency ω, close to the oscillation frequency Ω of the vessel. It is required to obtain the conditions for which the initial perturbation will increase with time.

In a system of coordinates rigidly attached to the vessel, the linear equations and boundary conditions for the velocity, $\mathbf{U}$, the pressure P and the elevation of the free surface Σ have the form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{e}_z s\Omega^2 \cos\Omega t = -\frac{1}{\rho}\nabla P - \mathbf{e}_z g + \nu\Delta\mathbf{U}, \quad \operatorname{div}\mathbf{U} = 0 \quad \text{in } D$$

$$\mathbf{U} = 0 \quad \text{on} \quad S$$

$$-P - \frac{\partial P}{\partial z}H + 2\nu\rho\frac{\partial U_z}{\partial z} = 0, \quad \nu\rho\left(\frac{\partial U_\xi}{\partial z} + \frac{\partial U_z}{\partial \xi}\right) = 0, \quad \xi = x, y$$

$$U_z = \frac{\partial H}{\partial t}(x, y, t) \quad \text{on } \Sigma$$

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Here ρ is the density, ν is the kinematic viscosity of the liquid, g is the acceleration due to gravity, $\mathbf{e}_z$ is the unit vector along the z axis, D is the region occupied by the liquid and S is the solid boundary of the region D. A Cartesian system of coordinates xyz is chosen so that the xy plane coincides with the surface Σ while the z axis and the unit vector $\mathbf{e}_z$ are directed vertically upwards.

We will introduce dimensionless variables taking the characteristic dimension d of the vessel as the unit of length and the quantity $1/\omega_0$ as the unit of time, where ω_0 is the least natural frequency of oscillations of an ideal two-layer liquid

$$\mathbf{U} = d\omega\mathbf{u}, \quad P = -g\rho z + d^2\omega_0^2\rho p - \rho s\Omega^2 z\cos\Omega t, \quad H = \eta d$$

We will represent the velocity vector $\mathbf{U}$ in the form of the sum of potential and vortex components

$$\mathbf{u} = -\nabla\varphi + \mathbf{v}$$

We will put

$$p = \partial\varphi/\partial t$$

Retaining the previous notation for all the dimensionless quantities, we obtain the following problem for the functions φ, $\mathbf{v}$ and η

$$\Delta\varphi = 0, \quad \frac{\partial\mathbf{v}}{\partial t} = \varepsilon^2\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } D \tag{1.1}$$

$$\nabla\varphi = \mathbf{v} \quad \text{on } S$$

$$F\frac{\partial\varphi}{\partial t} - \left(1 + \sqrt{\varepsilon}\gamma\cos\Omega t\right)\eta + \varepsilon^2 2F\left(\frac{\partial v_z}{\partial z} - \frac{\partial^2\varphi}{\partial z^2}\right) = 0 \tag{1.2}$$

$$\varepsilon\left(\frac{\partial v_\xi}{\partial z} + \frac{\partial v_z}{\partial\xi} - 2\frac{\partial^2\varphi}{\partial\xi\partial z}\right) = 0, \quad \xi = x, y; \quad v_z - \frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t}(x, y, t) \quad \text{on } \Sigma$$

where $F = \omega_0^2 d/g$, $\varepsilon^2 = \nu/(\omega_0 d^2)$, $\sqrt{\varepsilon}\gamma = s\Omega^2/g$, $\gamma = O(1)$. Assuming that the liquid has a low viscosity and the acceleration of the vessel is small compared with the acceleration due to gravity, we take $\varepsilon \ll 1$.

2. THE ZEROTH AND FIRST APPROXIMATIONS

The asymptotic solution of the singularly perturbed problem (1.1), (1.2) will be sought in the form

$$\begin{aligned} &\varphi = \Phi_0 + \sqrt{\varepsilon}\Phi_1 + \ldots \\ &\mathbf{v} = S\mathbf{v} + \Sigma\mathbf{v} \equiv S_0\mathbf{v} + \sqrt{\varepsilon}S_1\mathbf{v} + \ldots + \Sigma_0\mathbf{v} + \sqrt{\varepsilon}\Sigma_1\mathbf{v} + \ldots \\ &\eta = \eta_0 + \sqrt{\varepsilon}\eta_1 + \ldots \end{aligned} \tag{2.1}$$

Here Φ is the regular part of the asymptotic expansion, while $S\mathbf{v}$ and $\Sigma\mathbf{v}$ are the boundary parts, which exist only in the subregions D_S and D_Σ adjoining the surfaces S and Σ, respectively.

Using the idea of Krylov–Bogolyubov averaging method, we will assume that each of the functions in (2.1) depends on the spatial variable and on the so-called "slowly varying amplitude" C, the "fast phase" ψ and the "slow phase" θ.

The functions C, ψ and θ satisfy the relations

$$\begin{aligned} &\frac{dC}{dt} = \sqrt{\varepsilon}A_1(C,\theta) + \varepsilon A_2(C,\theta) + \ldots \\ &\frac{d\theta}{dt} = \Delta + \sqrt{\varepsilon}B_1(C,\theta) + \varepsilon B_2(C,\theta) + \ldots \\ &\theta = \psi - \Omega t, \quad \Delta = \omega - \Omega \end{aligned} \tag{2.2}$$

where $A_1(C, \theta), B_1(C, \theta), \ldots$ are periodic functions of θ, which, like the coefficients of expansions (2.1) are to be determined from problem (1.1), (1.2). We will also assume that $\Delta = O(\varepsilon)$.

Taking relations (2.2) into account, the partial derivatives with respect to t, for example, of the function φ, can be written in the form

$$\frac{\partial\varphi}{\partial t} = \omega\frac{\partial\Phi_0}{\partial\psi} + \sqrt{\varepsilon}\left(K_1(\Phi_0) + \omega\frac{\partial\Phi_1}{\partial\psi}\right) + \varepsilon\left(K_2(\Phi_0) + K_1(\Phi_1) + \omega\frac{\partial\Phi_2}{\partial\psi}\right) + \ldots$$

$$\frac{\partial^2\varphi}{\partial t^2} = \omega^2\frac{\partial^2\Phi_0}{\partial\psi^2} + \sqrt{\varepsilon}\left(2\omega L_1(\Phi_0) + \omega^2\frac{\partial^2\Phi_1}{\partial\psi^2}\right) +$$

$$+\varepsilon\left(\omega^2\frac{\partial^2\Phi_1}{\partial\psi^2} + 2\omega L_2(\Phi_0) + 2\omega L_1(\Phi_1) + A_1^2\frac{\partial^2\Phi_0}{\partial C^2} + 2A_1B_1\frac{\partial^2\Phi_0}{\partial C\partial\psi} + B_1^2\frac{\partial^2\Phi_0}{\partial\psi^2}\right) + \ldots \qquad (2.3)$$

$$K_n(\Phi_k) \equiv \frac{\partial\Phi_k}{\partial C}A_n + \frac{\partial\Phi_k}{\partial\psi}B_n, \quad k = 0,1; \quad n = 1,2$$

$$L_n(\Phi_k) \equiv \frac{\partial^2\Phi_k}{\partial\psi\partial C}A_n + \frac{\partial^2\Phi_k}{\partial\psi^2}B_n, \quad k = 0,1; \quad n = 1,2$$

Similar expansions exist for the boundary-layer functions.

We will introduce local orthogonal curvilinear coordinates s_1, s_2, s_3 into the region D, so that the surface $s_3 = 0$ coincides with S and $s_3 > 0$ in D. We will introduce Cartesian coordinates x, y, z into the region D_Σ so that the x axis coincides with the axis of the initial system of coordinates $y \in \Sigma$, and the z axis is directed into the region D.

We will require that the boundary-layer functions satisfy the following relations

$$S\mathbf{v} \to 0 \quad \text{as} \quad \sigma \to \infty, \quad \Sigma\mathbf{v} \to 0 \quad \text{as} \quad \zeta \to \infty \qquad (2.4)$$

Substituting expansions (2.1) and (2.3) into (1.1) and (1.2) and equating coefficients of like powers of ε we obtain a series of boundary-value problems for determining the coefficients of expansions (2.1).

The zeroth approximation, i.e. the functions which satisfy problem (1.1), (1.2) to within terms $O\left(\sqrt{\varepsilon}\right)$, can be found from equations similar to those considered previously [3]. We will only present the final result

$$\Phi_0 = Cf_0\cos\psi, \quad \eta_0 = -F\omega f_0\,|_{z=0}\sin\psi$$

$$S_0v_\sigma = 0, \quad \Sigma_0\mathbf{v} = 0$$

$$S_0v_l = H_l^{-1}\partial f_0/\partial s_l|_S C\exp(-\lambda\sigma)\cos(\psi - \lambda\sigma), \quad l = 1,2, \quad \lambda = (\omega/2)^{1/2}$$

Here f_0 is the eigenfunction of the problem

$$\Delta f_0 = 0 \quad \text{in } D$$

$$\frac{\partial f_0}{\partial n} = 0 \quad \text{on } S, \quad \frac{\partial f_0}{\partial z} = F\omega^2 f_0 \quad \text{on } \Sigma, \qquad (2.5)$$

corresponding to the eigenvalue $F\omega^2$, where n denotes the inward normal to the boundary of the region D while H_1, H_2, ($H_3 = 1$) are the Lamé coefficients of the system of coordinates s_1, s_2, s_3.

The problem for the function Φ_1 has the form

$$\Delta\Phi_1 = 0 \quad \text{in } D, \quad \frac{\partial\Phi_1}{\partial n} = 0 \quad \text{on } S$$

$$F\omega^2\frac{\partial^2\Phi_1}{\partial\psi^2} + \frac{\partial\Phi_1}{\partial z} = 2\omega FQ_1f_0 - \gamma\omega^2Cf_0\cos(2\psi - \theta) \quad \text{on } \Sigma \qquad (2.6)$$

$$Q_i = A_i\sin\psi + CB_i\cos\psi, \quad i = 1,2,\ldots$$

We will represent Φ_1 in the form

$$\Phi_1 = \Phi_1^{(1)} + \Phi_1^{(2)}$$

where $\Phi_1^{(1)}$, depends on ψ as $\sin\psi$ and $\cos\psi$, while $\Phi_1^{(2)}$, depends on ψ as $\sin 2\psi$ and $\cos 2\psi$. For $\Phi_1^{(1)}$, we obtain the problem

$$\Delta\Phi_1^{(1)} = 0 \quad \text{in } D$$
$$\frac{\partial\Phi_1^{(1)}}{\partial n} \quad \text{on } S, \quad \frac{\partial\Phi_1^{(1)}}{\partial z} - F\omega^2\Phi_1^{(1)} = 2\omega F Q_1 f_0 \quad \text{on } \Sigma \tag{2.7}$$

The condition for problem for (2.7) to be solvable has the form

$$\iint_\Sigma \left(f_0 \frac{\partial\Phi_1^{(1)}}{\partial z} - \Phi_1^{(1)} \frac{\partial f_0}{\partial z} \right) d\Sigma = 0 \tag{2.8}$$

Expressing the derivatives $\partial f_0/\partial z$, $\partial\Phi_1^{(1)}/\partial z$ on Σ from (2.5) and (2.7) and substituting them into (2.8), we obtain

$$2\omega F(A_1 \sin\psi + CB_1 \cos\psi)\iint_\Sigma f_0^2 d\Sigma = 0$$

Consequently, $A_1 = B_1 = 0$. Taking this into account, we can write the particular solution of problem (2.6) in the form

$$\Phi_1 = \tfrac{1}{3} C\gamma f_0 \cos(2\psi - \theta)$$

Using the explicit expression for Φ_1, we obtain

$$\eta_1 = -\tfrac{1}{2} F\omega C\gamma\left(\tfrac{1}{3}\sin(2\psi - \theta) + \sin\theta\right) f_0\big|_{z=0}$$

From the third equation of (1.1), considered in D_S and D_Σ, and conditions (2.4) we obtain

$$S_1 \upsilon_\sigma = \Sigma_1 \upsilon_\zeta = 0$$

From the second equation of (1.1), considered in D_S, the first condition of (1.2) and conditions (2.4) we obtain the problems for the functions $S_1\upsilon_1$ (l = 1, 2)

$$\omega\partial S_1\upsilon_l / \partial\psi = \partial^2 S_1\upsilon_l / \partial\sigma^2$$
$$S_1\upsilon_l = H_l^{-1}\partial\Phi_1 / \partial s_l \quad \text{on } S, \quad S_1\upsilon_l \to 0 \quad \text{as} \quad \sigma \to \infty$$

the solution of which, taking into account the explicit expression for Φ_1, can be written in the form

$$S_1\upsilon_l = \tfrac{1}{3} H_l^{-1}\gamma\partial f_0 / \partial s_l\big|_S C \exp\left(-\sqrt{\omega}\sigma\right)\cos\left(2\psi - \theta - \sqrt{\omega}\sigma\right)$$

From the second equation of (1.1), considered in D_Σ, the third condition of (1.2) and conditions (2.4) we obtain the problems for the functions $\Sigma_1\upsilon_\xi$ ($\xi = x, y$)

$$\omega\partial\Sigma_1\upsilon_\xi / \partial\psi = \partial^2\Sigma_1\upsilon_\xi / \partial\zeta^2$$
$$\partial\Sigma_1\upsilon_\xi / \partial\zeta = 0 \quad \text{on } \Sigma, \quad \Sigma_1\upsilon_\xi \to 0 \quad \text{as} \quad \zeta \to \infty$$

Consequently, $\Sigma_1\upsilon_\xi = 0$.

From the third equation of (1.1), considered in D_S, and conditions (2.4) we obtain

$$\Sigma_2\upsilon_\xi = 0$$

$$S_2 v_\sigma = (2\omega)^{-1/2} C G(s_1, s_2)\exp(-\lambda\sigma)(\sin(\psi - \lambda\sigma) + \cos(\psi - \lambda\sigma))$$

$$G(s_1, s_2) = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial s_1}\left(\frac{H_2}{H_1}\frac{\partial f_0}{\partial s_1}\right) + \frac{\partial f_0}{\partial s_2}\left(\frac{H_1}{H_2}\frac{\partial f_0}{\partial s_2}\right)\right]\Bigg|_{s_3=0}$$

3. APPROXIMATE FORMULAE FOR THE BOUNDARIES OF THE RESONANCE ZONES AND THE THRESHOLD AMPLITUDE

Considering the problem for the function Φ_2, we will represent this function in the form

$$\Phi_2 = \Phi_2^{(1)} + \Phi_2^{(2,3)}$$

where $\Phi_2^{(1)}$ depends on ψ as $\sin\psi$ and $\cos\psi$, while $\Phi_2^{(2,3)}$ depends on ψ as $\sin 2\psi$, $\cos 2\psi$, $\sin 3\psi$ and $\cos 3\psi$. For $\Phi_2^{(1)}$ we obtain the problem

$$\Delta\Phi_2^{(1)} = 0 \quad \text{in } D$$

$$\frac{\partial\Phi_1^{(1)}}{\partial n} = S_1 v_\sigma \quad \text{on } S, \quad \frac{\partial\Phi_2^{(1)}}{\partial z} - F\omega^2\Phi_2^{(1)} = 2\omega F Q_2 f_0 + \tag{3.1}$$

$$+\frac{\gamma^2\omega}{4}C\left[\left(\frac{2}{3} - \cos 2\theta\right)\cos\psi - \sin 2\theta \sin\psi\right] f_0 \quad \text{on } \Sigma$$

The condition for problem (3.1) to be solvable has the form

$$\iint_S f_0 \frac{\partial\Phi_2^{(1)}}{\partial n} dS - \iint_\Sigma \left(f_0 \frac{\partial\Phi_2^{(1)}}{\partial z} - \Phi_2^{(1)}\frac{\partial f_0}{\partial z}\right) d\Sigma = 0 \tag{3.2}$$

Expressing the derivatives $\partial f_0/\partial z$, $\partial\Phi_2^{(1)}/\partial z$ on Σ and $\partial\Phi_2^{(1)}/\partial n$ on S from (2.5) and (3.1) and substituting them into (3.2), we obtain after reduction

$$-2\sqrt{2}F\omega^{5/2}(\sin\psi + \cos\psi)I_S = 8(A_2\sin\psi + CB_2\cos\psi)I_\Sigma +$$

$$+C\gamma^2\omega\left[\left(\frac{2}{3} - \cos 2\theta\right)\cos\psi - \sin 2\theta\sin\psi\right]I_\Sigma \tag{3.3}$$

$$I_S = \iint_S (\nabla_2 f_0)^2 dS, \quad I_\Sigma = \iint_\Sigma \left(\frac{\partial f_0}{\partial z}\right)^2 d\Sigma$$

Equating the coefficients of $\sin\psi$ and $\cos\psi$ in (3.3) separately, we obtain the functions $A_2(C, \theta)$ and $B_2(C, \theta)$. We substitute these functions into (2.2), considered up to terms $O(\varepsilon)$. We have

$$\frac{dC}{dt} = -\varepsilon\alpha C + \varepsilon C\frac{\gamma^2\omega}{8}\sin 2\theta, \quad \frac{d\theta}{dt} = \Delta - \varepsilon\alpha + \varepsilon\frac{\gamma^2\omega}{8}\left(\cos 2\theta - \frac{2}{3}\right) \tag{3.4}$$

$$\alpha = 2^{-3/2}F_\omega^{5/2}I_S/I_\Sigma$$

To investigate the stability of the trivial solution $C = 0$, $\theta = \text{const}$, we reduce (3.4) to a linear system using the replacement $u = C\cos\theta$, $\upsilon = C\sin\theta$. We have

$$\frac{du}{dt} = -\varepsilon\alpha u - \varepsilon\left(\bar\Delta - \frac{\gamma^2\omega}{8}\right)\upsilon, \quad \frac{d\upsilon}{dt} = \varepsilon\left(\bar\Delta + \frac{\gamma^2\omega}{8}\right)u - \varepsilon\alpha\upsilon \tag{3.5}$$

$$\bar\Delta \equiv \Delta/\varepsilon - \alpha - \gamma^2\omega/12$$

The characteristic equation corresponding to system (3.5) has the solutions

$$\lambda_\pm = -\varepsilon\alpha \pm \left((\gamma^2\omega/8)^2 - \bar{\Delta}^2\right)^{1/2}$$

For the amplitude of the oscillations to increase it is necessary for the following inequality to be satisfied

$$(\gamma^2\omega/8)^2 - \bar{\Delta}^2 > \alpha^2$$

Hence, reverting to dimensional variables, we obtain

$$R_- < \frac{\Omega}{\omega} < R_+$$

$$R_\pm = 1 + \frac{\Delta\omega}{\omega} - \frac{1}{12}\left(\frac{s\Omega^2}{g}\right)^2 \pm \left[\frac{1}{64}\left(\frac{s\Omega^2}{g}\right)^4 - \left(\frac{\Delta\omega}{\omega}\right)^2\right]^{1/2} \tag{3.6}$$

where $\Delta\omega \equiv -\varepsilon\omega_0\alpha$ is the shift in the natural frequency of oscillations of an ideal liquid (see [3])

$$\Delta\omega = -2^{-3/2}\nu^{1/2}g^{-1}\omega^{5/2}I_S/I_\Sigma$$

Formula (3.6) gives expressions for the frequencies at which an increase in the amplitude of the surface waves in a liquid becomes possible in the region of harmonic resonance ($\omega = \Omega$) for a given amplitude s of the vessel oscillations. In this case the amplitude s itself should exceed a certain threshold value s_0, which can be found from the condition

$$(sg^{-1}\omega^2)^2/8 \geqslant |\Delta\omega|/\omega$$

Hence we obtain

$$s_0 = 2^{3/4}\nu^{1/4}g^{1/2}\omega^{-5/4}(I_S/I_\Sigma)^{1/2} \tag{3.7}$$

4. COMPARISON WITH EXPERIMENT

For a vessel in the form of a rectangular parallelepiped ($0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $-h \leqslant z \leqslant 0$) we have

$$I_S = \frac{ab}{2(2-\delta_{0m})\operatorname{sh}^2(k_{nm}h)}\left(\frac{\pi^2 \operatorname{sh}(2k_{nm}h)}{k_{nm}}\times\right.$$

$$\left.\times\left[\frac{n^2}{a^2}\left(\frac{2-\delta_{0m}}{b}+\frac{1}{a}\right)+\frac{m^2}{b^2}\left(\frac{2}{a}+\frac{1}{b}\right)\right]-2h\pi^2\left(\frac{n^2}{a^3}+\frac{m^2}{b^3}\right)+k_{nm}^2\right)$$

$$I_\Sigma = 2^{-2+\delta_{0m}}abk_{nm}^2;\quad k_{nm} = \pi\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)^{1/2},\quad n = 1,2,\ldots;\quad m = 0,1,\ldots,$$

where δ_{nm} is the Kronecker delta.

In order to check the theoretical results of Sections 1–3 on the equipment described previously in [7], we carried out a series of experiments to measure the width of the resonance zones of harmonic excitation of the second wave mode ($n = 2$) in a vertically oscillating rectangular vessel ($a = 50$ cm, $b = 4$ cm), filled with the water ($h = 15$ cm, $\nu = 0.01$ cm^2/s). The frequency range of the parametric excitation of the waves, by (3.6), is determined by the amplitude s and the frequency Ω of the vessel oscillations. We used the following procedure to estimate the limits of its range. We initially calculated the natural frequency of the second mode $\omega = 10.85$ s^{-1} and we established the oscillation frequency

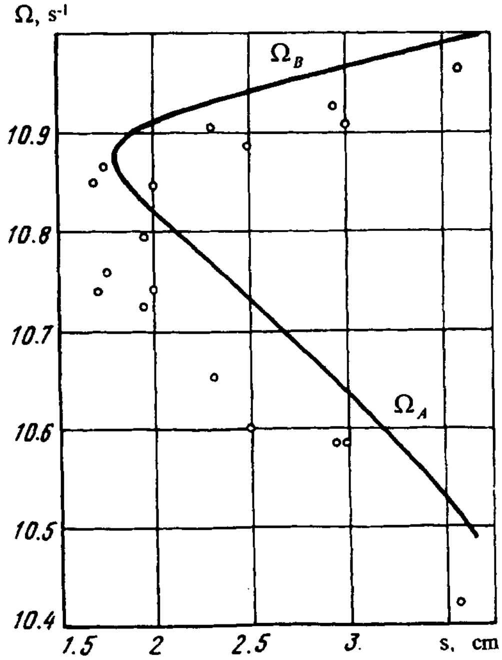

Fig. 1.

of the vessel $\Omega = \omega$ for the chosen amplitude s. After the oscillations reached a steady state the frequency Ω changed smoothly, so that the height of the wave decreased. This change continued to a value of Ω_B for which the wave amplitude was practically zero; Ω_B was taken as the limiting value. The other limit Ω_A of the range was found by discrete variation in small steps of the oscillation frequency of the vessel in the opposite direction, i.e. when the wave amplitude increased. The equipment was switched off for each new value of Ω. After complete cessation of the wave motions of the liquid, oscillations of frequency Ω were again applied to the vessel and the presence or absence of a wave buildup was recorded. If a steady state of the oscillations of the liquid was not achieved after 20 minutes, the corresponding value of the frequency was taken as the limiting frequency Ω_A.

For the second wave mode, the stability diagram is shown in the figure (the continuous curve is the boundary of the range of parametric excitation, calculated from (3.6), and the small circles represent experimental data).

It can be seen that there is a threshold oscillation amplitude of the vessel $s_0 = 1.69$ cm, below which, for any Ω, the free surface of the liquid remains unperturbed. By relation (3.7) the corresponding calculated value of the threshold amplitude $s_0 = 1.79$ cm.

Note that for the same values of a, b, h and n the threshold amplitude s_0 in the case of the fundamental resonance ($\omega = \Omega/2$) is equal to 0.04 cm (see [7]), i.e. one-fortieth of that for harmonic resonance.

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